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The path integral for arbitrary fermion sectors in supersymmetric quantum mechanics

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Abstract. I show how to obtain the kernel for arbitrary fermion sectors in supersymmetric quantum mechanics by applying standard coordinate space functional integral techniques to the coherent state fermionic path integral.

1. Introduction

The path integral has played an important role in the development of quantum field theory and other branches of physics [1-3]. While the theory of the bosonic path integral is well defined, the fermionic path integral is generally not so well defined. Usually one evaluates these fermionic path integrals formally by solving an eigenvalue problem using appropriate boundary conditions and using the eigenvalues to define a determinant.

In this paper I will show how standard coordinate space functional integral techniques applied to the coherent state fermionic path integral [4] leads to physically important results, such as the fermionic propagator, and to the evaluation of determinants without solving the corresponding eigenvalue problem†. I apply these results to supersymmetric quantum mechanics [6]. For supersymmetric quantum mechanics on a curved manifold (the supersymmetric nonlinear σ-model [7]) there is a four-fermion interaction term, and the path integral is no longer simply a determinant. Nevertheless, the functional integral techniques will be able to deal with this four-fermion term and, of equal importance, I will be able to write the short-time kernel which propagates solutions of the Schrödinger equation for *any* fermion sector, and not just for the zero and full fermion sectors which has been done previously [8-10].

The paper will be organized as follows. In section 2, I give a general discussion of the quadratic fermionic path integral. I then apply the results to supersymmetric quantum mechanics on a flat manifold. In section 3, I treat supersymmetric quantum mechanics on a curved manifold. I show how to obtain an approximate fermionic generating functional which allows one to deal with the four fermion term, and from which one can obtain the short time kernel and the 'top form' for the evaluation of the Witten index.

† For a similar discussion of the fermionic path integral see [5].

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2. The quadratic fermionic path integral

In this section I show how to apply standard coordinate space functional integral techniques [11] to the quadratic fermionic path integral. I then apply these results to supersymmetric quantum mechanics on a flat manifold, and find that one can propagate all of the supersymmetric states from the different fermionic sectors.

By way of introducing my notation I reproduce some standard results for the fermionic generating functional. Consider a general quadratic fermionic path integral with fermionic sources $\eta^{*\mu}(t)$ and $\eta_\nu(t)$. The fermionic generating functional is

$$Z[\eta^*, \eta] = \int \mathrm{D}[\psi^*]\mathrm{D}[\psi] \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t \left[\psi^{*\alpha}\left(\mathrm{i}\frac{\overset{\rightarrow}{\mathrm{d}}}{\mathrm{d}t}\delta^\beta_\alpha - A^\beta_\alpha - B^\beta_{\alpha\sigma}\dot{q}^\sigma\right)\psi_\beta + \eta^{*\alpha}\psi_\alpha + \psi^{*\alpha}\eta_\alpha\right]\right\}. \tag{2.1}$$

Consider a fermionic generating functional Z_0 which is the path integral (2.1) in which $A^\beta_\alpha = 0$ and $B^\beta_{\alpha\sigma} = 0$. By considering solutions of the resulting classical equations of motion which have retarded boundary conditions, i.e. $\psi_\alpha(t) = \mathrm{i}\int \mathrm{d}t'\, \theta(t-t')\eta_\alpha(t')$, then

$$Z[\eta^*, \eta]_0 = \exp\left(-\frac{1}{\hbar}\int \mathrm{d}t\,\mathrm{d}t'\, \eta^{*\mu}(t)\theta(t-t')\eta_\mu(t')\right). \tag{2.2}$$

It then follows that $Z[\eta^*, \eta]$ can be obtained from $Z[\eta^*, \eta]_0$ in the following fashion:

$$Z[\eta^*, \eta] = \exp\left[-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left(\tilde{\psi}^{*\alpha}[A^\beta_\alpha + B^\beta_{\alpha\sigma}\dot{q}^\sigma]\tilde{\psi}_\beta\right)\right] Z[\eta^*, \eta]_0 \tag{2.3}$$

where

$$\tilde{\psi}^{*\alpha}(t) = \left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta_\alpha(t)}\right) \qquad \text{and} \qquad \tilde{\psi}_\beta(t) = \left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\beta}(t)}\right).$$

Carrying out the expansion in (2.3) we obtain the general expression for the quadratic fermionic generating functional of (2.1)

$$Z[\eta^*, \eta] = \exp\left(-\frac{1}{\hbar}\int \mathrm{d}t\,\mathrm{d}t'\, \eta^{*\mu}(t)F^\nu_\mu(t-t')\eta_\nu(t')\right) Z[\eta^*, \eta]|_{\eta^*=\eta=0}. \tag{2.4}$$

In (2.4) $F^\nu_\mu(t-t')$ is the fermionic propagator and has the following formal expression†:

$$F^\nu_\mu(t-t') = \theta(t-t')T\left(\exp\int_{t'}^{t} \mathrm{d}t_1\{-\mathrm{i}A^\nu_\mu[q(t_1)] - \mathrm{i}B^\nu_{\mu\sigma}[q(t_1)]\dot{q}^\sigma(t_1)\}\right). \tag{2.5}$$

By setting $\eta^* = \eta = 0$ in (2.1) it follows that $Z[\eta^*, \eta]|_{\eta^*=\eta=0}$ can be expressed as a determinant (with retarded boundary conditions) which is

$$\begin{aligned} Z[\eta^*, \eta]|_{\eta^*=\eta=0} &= \det\left|\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta^\beta_\alpha - A^\beta_\alpha[q(t)] - B^\beta_{\alpha\sigma}[q(t)]\dot{q}^\sigma\right)\delta(t-t')\right| \\ &= \exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left(\frac{\hbar}{2}A^\alpha_\alpha + \frac{\hbar}{2}B^\alpha_{\alpha\sigma}\dot{q}^\sigma + \frac{\hbar^2}{8}g^{\mu\nu}B^\beta_{\alpha\mu}B^\alpha_{\beta\nu}\right)\right]. \end{aligned} \tag{2.6}$$

† I am using the notation $\mathrm{e}^{A^\mu_\nu} = [\mathrm{e}^A]^\mu_\nu$ if any indices remain uncontracted.

This shows that one can calculate determinants without solving the corresponding formal eigenvalue problem. It should be noted that in obtaining (2.6) I used the fact that for quantum paths the correlation $\langle 0|T\dot{q}^{\mu}(t)\dot{q}^{\nu}(t')|0\rangle = \mathrm{i}\hbar g^{\mu\nu}[q(t)]\delta(t-t')$, and also the following two identities: $\theta(t-t')\theta(t'-t)=0$, and $\theta(0)=\frac{1}{2}$.

Having obtained the generating functional for a general quadratic fermionic path integral, I will use it to propagate states in the different fermionic sectors of supersymmetric quantum mechanics. Before doing this I will review supersymmetric quantum mechanics on a flat N-dimensional manifold [6] and find the Schrödinger equation for the supersymmetric states. The Lagrangian for supersymmetric quantum mechanics on a flat N-dimensional manifold described in Cartesian coordinates is

$$L=\tfrac{1}{2}\eta_{ab}\dot{q}^a\dot{q}^b-\tfrac{1}{2}\eta^{ab}(\partial_a V)(\partial_b V)+\psi^{*a}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta_a^b-\eta^{bc}\partial_a\partial_c V\right)\psi_b. \tag{2.7}$$

Upon quantization, the supersymmetric Hamiltonian is

$$H=\tfrac{1}{2}\eta^{ab}P_aP_b+\tfrac{1}{2}\eta^{ab}(\partial_a V)(\partial_b V)+\eta^{bc}(\partial_a\partial_c V)(\tfrac{1}{2}\hat{\psi}^{*a}\hat{\psi}_b-\tfrac{1}{2}\hat{\psi}_b\hat{\psi}^{*a}). \tag{2.8}$$

The supersymmetric states on which the Hamiltonian acts are of the form $\Psi[q(t)]=A_{\alpha_1\ldots\alpha_p}[q(t)]\hat{\psi}^{*\alpha_1}\ldots\hat{\psi}^{*\alpha_p}|0\rangle$, where $\hat{\psi}^{*\alpha_1}$ and $\hat{\psi}_{\alpha_1}$ are creation and annihilation operators and $|0\rangle$ is the fermionic vacuum for which $\hat{\psi}_{\alpha_1}|0\rangle=0$. If p is even (odd) then the supersymmetric state is a boson (fermion). The Schrödinger equation for the supersymmetric states is

$$\begin{aligned}\mathrm{i}\hbar\partial_t A_{\alpha_1\ldots\alpha_p}&=\left(-\frac{\hbar^2}{2}\eta^{ab}\partial_a\partial_b+\tfrac{1}{2}\eta^{ab}(\partial_a V)(\partial_b V)-\frac{\hbar}{2}\eta^{ab}(\partial_a\partial_b V)\right)A_{\alpha_1\ldots\alpha_p}\\&\quad+\hbar\sum_{\lambda=1}^{p}\eta^{ab}(\partial_{\alpha_\lambda}\partial_b V)A_{\alpha_1\ldots\alpha_{\lambda-1}a\alpha_{\lambda+1}\ldots\alpha_p}.\end{aligned} \tag{2.9}$$

Note that for the zero fermion sector the Hamiltonian is diagonal in tensor indices and the propagator for the zero fermion sector is

$$\int \mathrm{D}[q]\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left(\tfrac{1}{2}\eta_{ab}\dot{q}^a\dot{q}^b-\tfrac{1}{2}\eta^{ab}(\partial_a V)(\partial_b V)+\frac{\hbar}{2}\eta^{ab}(\partial_a\partial_b V)\right)\right]. \tag{2.10}$$

For the general p-fermion sector the Hamiltonian is no longer diagonal and my motivation was to find the generalization of (2.10) which will allow one to propagate supersymmetric states from any fermionic sector. The kernel for the propagation of supersymmetric states can be obtained from the fermionic generating functional as follows:

$$A_{\alpha_1\ldots\alpha_p}[q,t_f]=\int \mathrm{d}q_0\,K^{\beta_1\ldots\beta_1}_{\alpha_1\ldots\alpha_p}(q,t_f;q_0,t_0)A_{\beta_1\ldots\beta_p}[q_0,t_0]. \tag{2.11}$$

In the above expression the kernel $K^{\beta_1\ldots\beta_1}_{\alpha_1\ldots\alpha_p}(q,t_f;q_0,t_0)$ is the path integral

$$K^{\beta_1\ldots\beta_1}_{\alpha_1\ldots\alpha_p}(q,t_f;q_0,t_0)=\int_{q_0}^{q}\mathrm{D}[q]M^{\beta_1\ldots\beta_1}_{\alpha_1\ldots\alpha_p}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_f}\mathrm{d}t\,L_{\mathrm{B}}\right] \tag{2.12}$$

where L_{B} is the bosonic part of the supersymmetric Lagrangian, and

$$\begin{aligned}M^{\beta_1\ldots\beta_1}_{\alpha_1\ldots\alpha_p}&=\frac{1}{p!\,\hbar^p}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta_{\beta_p(t_0)}}\right)\ldots\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta_{\beta_1(t_0)}}\right)\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\alpha_1}(t_f)}\right)\\&\quad\ldots\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\alpha_p}(t_f)}\right)Z[\eta^*,\eta]|_{\eta^*=\eta=0}.\end{aligned} \tag{2.13}$$

Thus once one has the fermionic generating functional for supersymmetric quantum mechanics one can use (2.11)-(2.13) to propagate solutions of the Schrödinger equation for any fermionic sector. Note that for the quadratic fermionic path integral there is no self-interaction of the fermionic propagators and hence one has the following factorization:

$$M^{\beta_1 \dots \beta_1}_{\alpha_1 \dots \alpha_p} A_{\beta_1 \dots \beta_p} = Z[\eta^*, \eta]|_{\eta^* = \eta = 0} F^{\beta_1}_{\alpha_1}(t_f - t_0) \dots F^{\beta_p}_{\alpha_p}(t_f - t_0) A_{\beta_1 \dots \beta_p}. \tag{2.14}$$

One can interpret the fermionic propagator $F^{\beta_1}_{\alpha_1}(t_f - t_0)$ as propagating the discrete index β_1 at time t_0 to the discrete index α_1 at time t_f.

Returning to the supersymmetric quantum mechanics of (2.7), the fermionic generating functional is

$$Z[\eta^*, \eta] = \exp\left(-\frac{1}{\hbar}\int \mathrm{d}t\, \mathrm{d}t'\, \eta^{*a}(t) F^b_a(t - t')\eta_b(t')\right) \times \exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left(\frac{\hbar}{2}\eta^{ab}(\partial_a\partial_b V)\right)\right] \tag{2.15}$$

where the fermionic propagator is

$$F^b_a(t - t') = \theta(t - t')T\left[\exp\int_{t'}^{t} \mathrm{d}t_1\{-\mathrm{i}\eta^{bc}(\partial_a\partial_c V)\}\right]. \tag{2.16}$$

To show that (2.11) does in fact lead to the Schrödinger equation, perform the standard technique of expanding the path integral to first order in time for an infinitesimal time step [1, 12]. I will given an explicit example; for ease of calculation I choose the one-fermion sector $A_\sigma[q(t)]$. Expanding to first order in time

$$\begin{aligned} A_\sigma[q, t] = \int \frac{\mathrm{d}^N \Delta q}{(2\pi \mathrm{i}\hbar\Delta t)^{N/2}} &\exp\left[\frac{\mathrm{i}}{\hbar}\Delta t\left(\frac{1}{2}\eta_{ab}\frac{\Delta q^a}{\Delta t}\frac{\Delta q^b}{\Delta t}\right)\right] \\ &\times\left(1 - \frac{\mathrm{i}}{2\hbar}\Delta t\eta^{ab}(\partial_a V)(\partial_b V) + \frac{\mathrm{i}}{2}\Delta t\eta^{ab}(\partial_a\partial_b V)\right) \\ &\times(\delta^\tau_\sigma - \mathrm{i}\eta^{\tau c}\Delta t(\partial_\sigma\partial_c V))\, A_\tau[q_0, t_0] \end{aligned} \tag{2.17}$$

which can be shown to satisfy the Schrödinger equation (2.9) when the results of appendix A are used.

I end this section by considering the supersymmetric quantum mechanics given by the Lagrangian

$$L = \tfrac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + \psi^{*\mu}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta^\nu_\mu - \mathrm{i}\Gamma^\nu_{\mu\alpha}\dot{q}^\alpha\right)\psi_\nu. \tag{2.18}$$

This is equivalent to the supersymmetric system of (2.7) in which the superpotential has been set to zero, and a curvilinear coordinate system is used instead of the usual Cartesian coordinates. From (2.5) and (2.6)

$$F^\nu_\mu(t - t') = \theta(t - t')T\left(\exp\int_{t'}^{t} \mathrm{d}t_1\{\Gamma^\nu_{\mu\alpha}[q(t_1)]\dot{q}^\alpha(t_1)\}\right) \tag{2.19}$$

and

$$Z[\eta^*, \eta]|_{\eta^* = \eta = 0} = \exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left(\frac{\mathrm{i}\hbar}{2}\Gamma^\alpha_{\mu\alpha}\dot{q}^\mu - \frac{\hbar^2}{8}g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta}\right)\right]. \tag{2.20}$$

By expanding to first order in time (using the results of appendix A) it can be shown that (2.11) leads to the Schrödinger equation

$$\mathrm{i}\hbar\partial_t A_{\alpha_1\ldots\alpha_p} = -\frac{\hbar^2}{2} g^{\mu\nu}\nabla_\mu\nabla_\nu A_{\alpha_1\ldots\alpha_p}. \tag{2.21}$$

3. The quartic fermionic path integral

The supersymmetric nonlinear σ-model in $(0-1)$ dimensions [7] is an example of supersymmetric quantum mechanics which contains a four-fermion interaction term. Therefore the fermionic propagators will no longer factorize as in the quadratic theory, and one can no longer equate the fermionic path integral with a determinant, as is usually done in the quadratic case.

In this section I show how to develop the standard coordinate space functional technique of section 2, to deal with the quartic fermionic path integral. I will be able to find a short time approximation for the fermionic generating functional, from which one can obtain solutions of the Schrödinger equation in any fermion sector, by use of the formalism of section 2. I will also show that one can obtain the Witten index from this short-time fermionic generating functional.

To start with I give the Lagrangian for the supersymmetric nonlinear σ-model and outline how to obtain the appropriate fermionic generating functional $Z_\sigma[\eta^*, \eta]$. The Lagrangian for the supersymmetric nonlinear σ-model is

$$L = \tfrac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + \psi^{*\mu}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta^\nu_\mu - \mathrm{i}\Gamma^\nu_{\mu\alpha}\dot{q}^\alpha\right)\psi_\nu + \tfrac{1}{2}R_\alpha{}^\beta{}_\gamma{}^\delta\psi^{*\alpha}\psi_\beta\psi^{*\gamma}\psi_\delta. \tag{3.1}$$

The fermionic generating functional $Z_\sigma[\eta^*, \eta]$ can be obtained from the fermionic generating functional for supersymmetric quantum mechanics on a flat manifold as follows:

$$Z_\sigma[\eta^*, \eta] = \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left(\tfrac{1}{2}R_\alpha{}^\beta{}_\gamma{}^\delta\tilde{\psi}^{*\alpha}\tilde{\psi}_\beta\tilde{\psi}^{*\gamma}\tilde{\psi}_\delta\right)\right)Z[\eta^*, \eta] \tag{3.2}$$

where

$$\tilde{\psi}^{*\alpha}(t) = \left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta_\alpha(t)}\right) \qquad \text{and} \qquad \tilde{\psi}_\beta(t) = \left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\beta}(t)}\right)$$

and $Z[\eta^*, \eta]$ has contributions from (2.19) and (2.20). In a straightforward manner

$$\begin{aligned} Z_\sigma[\eta^*, \eta]\big|_{\eta^*=\eta=0} &= \int \mathrm{D}[\psi^*]\mathrm{D}[\psi]\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\, L_\mathrm{F}\right] \\ &= \exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left(\frac{\mathrm{i}\hbar}{2}\Gamma^\alpha_{\mu\alpha}\dot{q}^\mu - \frac{\hbar^2}{8}g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} + \frac{\hbar^2}{8}R\right)\right] \end{aligned} \tag{3.3}$$

where L_F is the fermionic part of the supersymmetric Lagrangian equation (3.1). The kernel for the propagation of the zero-fermion sector is given by

$$\int \mathrm{D}[q]\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left(\tfrac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\right)\right]Z_\sigma[\eta^*, \eta]\big|_{\eta^*=\eta=0}. \tag{3.4}$$

This is exactly the kernel for a non-relativistic point particle moving on a curved manifold [13]. Therefore the zero fermion sector for the nonlinear σ-model is a non-relativistic point particle on a curved background.

The fermionic generating functional of (3.2) can be expressed as the product of $2j$-point generating functionals (see appendix B for notation).

$$Z_\sigma[\eta^*, \eta] = \prod_{j=1}^{\infty} Z_\sigma^{(2j)}[\eta^*, \eta]. \tag{3.5}$$

In appendix B I outline how to obtain the different $2j$-point generating functionals by use of the recurrence relations satisfied by the Green functions. In particular it is argued that one can obtain a physical fermionic generating functional by finding the first two terms in this product. The argument is as follows: in the Schrödinger picture we require that the kernel propagate the solutions of the Schrödinger equation. It is well known that in the short-time limit one need only keep terms up to order $O(\Delta t)$ in our kernel. Now our generating functional $Z_\sigma^{(4)}[\eta^*, \eta]$ has lowest order $O(\Delta t)$ (after the fermionic sources have been removed by differentiation) and thus contributes to the short-time kernel. For $Z_\sigma^{(2j)}[\eta^*, \eta]$, where $j>2$, these $2j$-point generating functionals will have lowest order $O(\Delta t^k)$ where $k \geqslant 2$, and thus they do not contribute to the short-time kernel. Hence the generating functional can be approximated by the truncated product

$$Z_\sigma[\eta^*, \eta] = Z_\sigma^{(2)}[\eta^*, \eta] Z_\sigma^{(4)}[\eta^*, \eta] \tag{3.6}$$

where

$$Z_\sigma^{(2)}[\eta^*, \eta] = \exp\left(-\frac{1}{\hbar}\int \mathrm{d}t\,\mathrm{d}t'\,\eta^{*\mu}(t) G_\mu^\nu(t-t')\eta_\nu(t')\right) Z_\sigma[\eta^*, \eta]\big|_{\eta^*=\eta=0} \tag{3.7}$$

and

$$Z_\sigma^{(4)}[\eta^*, \eta] = \exp\left(-\frac{\mathrm{i}}{4\hbar}\int \mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4\,\eta^{*\mu}(t_1)G_\mu^\alpha(t_1-t)\eta^{*\nu}(t_2)G_\nu^\gamma(t_2-t) \right.$$
$$\left. \times D_{\alpha\gamma}{}^{\beta\delta}[q(t), q(t')] G_\beta^\omega(t'-t_3)\eta_\omega(t_3) G_\delta^\sigma(t'-t_4)\eta_\sigma(t_4)\right). \tag{3.8}$$

It is interesting to note that the identity $\theta(t-t')\theta(t'-t)=0$ leads to considerable simplification of the Feynman diagrams. For example the 'dressed' fermionic propagator G_ν^μ for the nonlinear σ-model can be expressed as the sum of propagators F_ν^μ of the quadratic theory given by (2.19) in which one sums over only one infinite series of loops (see figure 1).

The infinite series can be summed and the fermionic propagator is

$$G_\mu^\nu(t-t') = \theta(t-t') T\left[\exp \int_{t'}^{t} \mathrm{d}t_1 \left(\Gamma^\nu_{\mu\alpha}[q(t_1)]\dot{q}^\alpha(t_1) - \frac{\mathrm{i}\hbar}{2} R^\mu_\nu[q(t_1)]\right)\right]. \tag{3.9}$$

Also the 'dressed' four-fermion vertex $D_{\alpha\gamma}{}^{\beta\delta}[q(t), q(t')]$ (which is given in (B8) of appendix B) contains a sum over only one infinite series of loops of the form shown in figure 2.

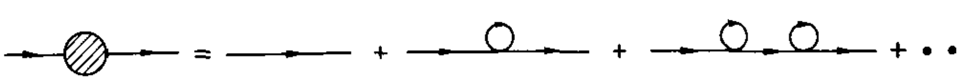

Figure 1. Fermionic propagator.

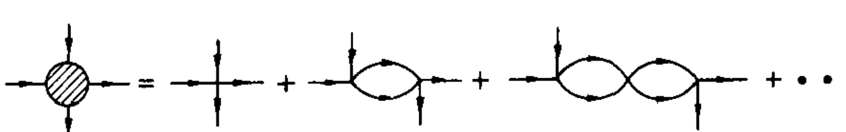

Figure 2. Four-fermion vertex.

Using the machinery developed in section 2, and the fermionic generating functional $Z_\sigma[\eta^*, \eta] = Z_\sigma^{(2)}[\eta^*, \eta] Z_\sigma^{(4)}[\eta^*, \eta]$ the Schrödinger equation for the supersymmetric wavefunctions is found to be equivalent to the Laplacian acting on differential p-forms, which is the geometrical interpretation of supersymmetric quantum mechanics [7]. That is, the Schrödinger equation is

$$\mathrm{i}\hbar \partial_t A_{\alpha_1 \ldots \alpha_p} = \frac{\hbar^2}{2}(d+\delta)^2 A_{\alpha_1 \ldots \alpha_p} \tag{3.10}$$

where $(d+\delta)^2$ is the Laplacian acting on differential p-forms given by [14]:

$$\begin{aligned}(d+\delta)^2 A_{\alpha_1 \ldots \alpha_p} &= -g^{\mu\nu} \nabla_\mu \nabla_\nu A_{\alpha_1 \ldots \alpha_p} + \sum_{\lambda=1}^{p} R^{\mu}_{\sigma_\lambda} A_{\alpha_1 \ldots \alpha_{\lambda-1} \mu \alpha_{\lambda+1} \ldots \alpha_p} \\ &\quad + \frac{1}{2} \sum_{\lambda=1}^{p} \sum_{\tau=1}^{p} R^{\mu\nu}{}_{\alpha_\lambda \alpha_\tau} A_{\alpha_1 \ldots \alpha_{\lambda-1} \mu \alpha_{\lambda+1} \ldots \alpha_{\tau-1} \nu \alpha_{\tau+1} \ldots \alpha_p}.\end{aligned} \tag{3.11}$$

In [15, 16] it was shown that the Witten index $\mathrm{Tr}(-1)^{\mathrm{F}}$ can be regularized by a supersymmetric path integral in imaginary time, where both the bosonic and fermionic path integrals have periodic boundary conditions. I will show that for a two-dimensional closed compact manifold, the truncated fermionic generating functional $Z_\sigma[\eta^*, \eta] = Z_\sigma^{(2)}[\eta^*, \eta] Z_\sigma^{(4)}[\eta^*, \eta]$ can be used to regularize $\mathrm{Tr}(-1)^{\mathrm{F}}$. Thus

$$\mathrm{Tr}(-1)^{\mathrm{F}} = \int \mathrm{D}[q][M - M^{\alpha}_{\alpha} + M^{\alpha_1 \alpha_2}_{\alpha_1 \alpha_2}] \exp\left(-\frac{1}{\hbar}\int \mathrm{d}t(\tfrac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu)\right). \tag{3.12}$$

In the above expression the bosonic path integral has periodic boundary conditions, and is over imaginary time. Also note that I have explicitly taken the trace for each of the fermionic sectors; the minus sign for the one fermion sector is due to the operator $(-1)^{\mathrm{F}}$. One can use the standard arguments to find the major contribution to the imaginary time path integral with periodic boundary conditions [1, 2]. The major contributions will come from the stationary paths. Thus

$$\mathrm{Tr}(-1)^{\mathrm{F}} = \int \frac{\mathrm{d}^2 q\, g^{1/2}}{(2\pi \mathrm{i} \hbar \Delta t)} [\tilde{M} - \tilde{M}^{\alpha}_{\alpha} + \tilde{M}^{\alpha_1 \alpha_2}_{\alpha_1 \alpha_2}]. \tag{3.13}$$

In (3.13) the tildes indicate that $\dot{q}^\alpha$ has been set to zero in the fermionic propagator $G^{\nu}_{\mu}(t-t')$ of (3.9) in accordance with the above result that the major contributions are obtained from the stationary paths. Now $\mathrm{Tr}(-1)^{\mathrm{F}}$ is a topological invariant; hence in (3.13) one can perform a short-time expansion and look for the time-independent term, giving

$$\mathrm{Tr}(-1)^{\mathrm{F}} = \frac{1}{8\pi} \varepsilon^{\alpha_1 \alpha_2} \varepsilon_{\beta_1 \beta_2} \int \mathrm{d}^2 q\, g^{1/2} R_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} \tag{3.14}$$

which (up to a sign) is the Gauss-Bonnet-Chern-Avez formula [17] for the Euler character of a closed compact two-dimensional manifold.

To generalize the above calculation from two to N dimensions one has

$$\mathrm{Tr}(-1)^{\mathrm{F}} = \int \frac{\mathrm{d}^N q\, g^{1/2}}{(2\pi \mathrm{i} \hbar \Delta t)^{N/2}} [\tilde{M} - \tilde{M}^{\alpha}_{\alpha} + \tilde{M}^{\alpha_1 \alpha_2}_{\alpha_1 \alpha_2} - \tilde{M}^{\alpha_1 \alpha_2 \alpha_3}_{\alpha_1 \alpha_2 \alpha_3} + \ldots + \tilde{M}^{\alpha_1 \ldots \alpha_N}_{\alpha_1 \ldots \alpha_N}]. \tag{3.15}$$

Of course to show that (3.15) does indeed give the correct Euler character one would have to calculate the different $2j$-point generating functionals and show that to order

$O(\Delta t^{N/2})$ that all the terms cancel except for the topological Euler character. This would be very tedious, but would constitute a proof of the index theorem for a De Rham complex.

Instead of the above generalization to N-dimensional manifolds I would like to present the following heuristic generalization. Note that $Z_\sigma^{(4)}[\eta^*, \eta]$ contains the 'top form', to be explained below. So $Z_\sigma^{(4)\mathrm{Top}}[\eta^*, \eta]$ will be the only fermionic generating functional required to obtain the Euler character. The fermionic generating functionals can be expanded in powers of the short time Δt and in powers of $\hbar$. However $\mathrm{Tr}(-1)^{\mathrm{F}}$ is a topological invariant and is independent of both Δt and $\hbar$, hence one can choose to expand the fermionic generating functionals in the short time Δt keeping terms up to order $O(\Delta t)$ (after the fermionic sources have been removed by differentiation). Thus as before the infinite product becomes truncated and $\tilde{Z}_\sigma[\eta^*, \eta] = \tilde{Z}_\sigma^{(2)}[\eta^*, \eta]\tilde{Z}_\sigma^{(4)}[\eta^*, \eta]$. Next perform a small $\hbar$ expansion to obtain

$$\tilde{Z}_\sigma^{(2)}[\eta^*, \eta] \to \exp\left(-\frac{1}{\hbar}\int \mathrm{d}t\,\mathrm{d}t'\,\eta^{*\mu}(t)\theta(t-t')\eta_\mu(t')\right) \tag{3.16}$$

and

$$\begin{aligned}\tilde{Z}_\sigma^{(4)}[\eta^*, \eta] &\to Z_\sigma^{(4)\mathrm{Top}}[\eta^*, \eta]\\ &= \exp\left(-\frac{\mathrm{i}}{4\hbar}\int \mathrm{d}t\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4\,\eta^{*\alpha}(t_1)\theta(t_1-t)\eta^{*\gamma}(t_2)\theta(t_2-t)\right.\\ &\quad\left.\times R_{\alpha\gamma}{}^{\beta\delta}[q(t)]\theta(t'-t_3)\eta_\beta(t_3)\theta(t'-t_4)\eta_\delta(t_4)\right).\end{aligned} \tag{3.17}$$

It then follows that

$$\begin{aligned}\mathrm{Tr}(-1)^{\mathrm{F}} = \int \frac{\mathrm{d}^N q\, g^{1/2}}{(2\pi\mathrm{i}\hbar\Delta t)^{N/2}}\frac{1}{N!\,\hbar^N}\left[\prod_{\alpha=1}^{N}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta_\alpha(t_0)}\right)\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\alpha}(t_f)}\right)\right]\\ \times Z_\sigma^{(4)\mathrm{Top}}[\eta^*, \eta]\big|_{\eta^*=\eta=0}\end{aligned} \tag{3.18}$$

where one performs a short-time expansion and look for the term independent of both Δt and $\hbar$. This leads to the result $\mathrm{Tr}(-1)^{\mathrm{F}} = (-1)^m\chi(M^{2m})$, where $N = 2m$ is the dimension of the closed compact manifold, and $\chi(M^{2m})$ is the Euler character, given by the Gauss-Bonnet-Chern-Avez formula

$$\begin{aligned}\chi(M^{2m}) = \frac{(-1)^m}{(4\pi)^m m!2^m}\int \mathrm{d(vol)}\,\varepsilon^{\alpha_1\gamma_1\ldots\alpha_m\gamma_m}\varepsilon_{\beta_1\delta_1\ldots\beta_m\delta_m}\\ \times R_{\alpha_1\gamma_1}{}^{\beta_1\delta_1}\ldots R_{\alpha_m\gamma_m}{}^{\beta_m\delta_m}.\end{aligned} \tag{3.19}$$

In conclusion I have shown how the fermionic generating functional obtained from the fermionic path integral can give the physically important quantities. When applied to supersymmetric quantum mechanics on a curved manifold one can deal with the four fermion interaction term and obtain the fermionic propagator, which leads to the kernel for any fermion sector.

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Appendix A

In this appendix I shall give the Schrödinger equation associated with the path integral. I follow the standard procedure [1, 12] for expanding the path integral to first order in time for an infinitesimal time step. It is known that the path integral is the continuum limit of the mid-point discrete path integral, and that the mid-point discrete path integral corresponds to a Weyl ordered Hamiltonian [18]. Thus expanding a path integral to order $O(\Delta t)$ gives the general form

$$\Psi[q,t]=\int\frac{\mathrm{d}^N\Delta q\, g^{1/2}(\bar{q})}{(2\pi\mathrm{i}\hbar\Delta t)^{N/2}}\exp\left[\frac{\mathrm{i}}{\hbar}\Delta t\left(\frac{1}{2}g_{\mu\nu}(\bar{q})\frac{\Delta q^\mu}{\Delta t}\frac{\Delta q^\nu}{\Delta t}\right)\right]$$
$$\times\left(1+\frac{\mathrm{i}\alpha}{\hbar}A_\mu(\bar{q})\Delta q^\mu-\frac{\mathrm{i}\Delta t}{\hbar}\beta C\right)\Psi[q_0,t_0] \qquad \text{(A1)}$$

where as usual $\Delta t=t-t_0$, $\Delta q^\mu=q^\mu-q_0^\mu$, and $A_\mu(\bar{q})$ means that the function A_μ is evaluated at the mid-point $\frac{1}{2}q+\frac{1}{2}q_0$. By a standard expansion technique [1, 12] the expression in (A1) is seen to lead to the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\Psi=H\Psi \qquad \text{(A2)}$$

where the Hamiltonian is Weyl ordered and has the form

$$H=[\tfrac{1}{2}g^{\mu\nu}P_\mu P_\nu]w+\alpha[P_\mu g^{\mu\nu}A_\nu]_W+\beta C. \qquad \text{(A3)}$$

$[\;]_W$ is used to represent the Weyl ordering of operators and, in particular,

$$\begin{aligned}[\tfrac{1}{2}g^{\mu\nu}P_\mu P_\nu]_W&=\tfrac{1}{8}P_\mu P_\nu g^{\mu\nu}+\tfrac{1}{8}g^{\mu\nu}P_\mu P_\nu+\tfrac{1}{4}P_\mu g^{\mu\nu}P_\nu\\ [P_\mu g^{\mu\nu}A_\nu]_W&=\tfrac{1}{2}P_\mu g^{\mu\nu}A_\nu+\tfrac{1}{2}g^{\mu\nu}A_\nu P_\mu.\end{aligned} \qquad \text{(A4)}$$

A useful application of the above, which is used many times in the paper, is the expression of the form

$$\Psi[q,t]=\int\frac{\mathrm{d}^N\Delta q\, g^{1/2}(\bar{q})}{(2\pi\mathrm{i}\hbar\Delta t)^{N/2}}\exp\left[\frac{\mathrm{i}}{\hbar}\Delta t\left(\tfrac{1}{2}g_{\mu\nu}(\bar{q})\frac{\Delta q^\mu}{\Delta t}\frac{\Delta q^\nu}{\Delta t}\right)\right]$$
$$\times\left[1-\frac{\mathrm{i}}{\hbar}\left(-\frac{\mathrm{i}\hbar}{2}\right)\Gamma^\gamma_{\mu\gamma}(\bar{q})\Delta q^\mu-\frac{\mathrm{i}\Delta t}{\hbar}\left(-\frac{\hbar^2}{8}\right)g^{\mu\nu}\Gamma^\gamma_{\mu\gamma}\Gamma^\beta_{\nu\beta}\right.$$
$$\left.-\frac{\mathrm{i}\Delta t}{\hbar}\left(\frac{\hbar^2}{8}\right)g^{\mu\nu}\Gamma^\beta_{\mu\gamma}\Gamma^\gamma_{\nu\beta}\right]\Psi[q_0,t_0]. \qquad \text{(A5)}$$

This leads to the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\Psi=\tfrac{1}{2}g^{\mu\nu}P_\mu P_\nu A+\frac{\mathrm{i}\hbar}{2}g^{\mu\nu}\Gamma^\lambda_{\mu\nu}P_\lambda\Psi \qquad \text{(A6)}$$

where the identity

$$\frac{\hbar^2}{8}[g^{\mu\nu}\Gamma^\beta_{\nu\alpha}\Gamma^\alpha_{\nu\beta}-R]=\frac{\hbar^2}{8}[(\partial_\mu\partial_\nu g^{\mu\nu})+g^{\mu\nu}\Gamma^\alpha_{\mu\alpha}\Gamma^\beta_{\nu\beta}+2(\partial_\mu g^{\mu\nu}\Gamma^\alpha_{\nu\alpha})] \qquad \text{(A7)}$$

has been used.

Appendix B

In this appendix I will derive the fermionic generating functional from which one can obtain the kernel for the propagation of solutions to the Schrödinger equation. For supersymmetric quantum mechanics on a curved manifold there is a quartic fermion interaction term, therefore the fermionic propagators will no longer factorize as they do for the Gaussian theory.

One can obtain the appropriate generating functional by considering the classical equations of motion satisfied by the generating functional, i.e. the Dyson–Schwinger equation. I do not solve for the generating functional which satisfies the Dyson–Schwinger equation, but instead obtain an approximate generating functional by solving the first two recurrence relations for Green functions which are obtained from the Dyson–Schwinger equation by differentiating with respect to η^* and η at $\eta^* = \eta = 0$. I will show that up to first order in time (all that is required to for the short time propagation of solutions to the Schrödinger equation) only the four-point fermionic generating functional is required.

The first recurrence relation for the fermionic Green function is

$$\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta_\tau(t_0)}\right)\left[\left(\frac{\delta S}{\delta\psi^{*\mu}(t)}\right)_{\mathrm{op}}+\eta^{*\mu}(t)\right]Z_\sigma[\eta^*,\eta]|_{\eta^*=\eta=0}=0. \tag{B1}$$

Where in $(\delta S/\delta\psi^{*\mu}(t))_{\mathrm{op}}$ the fermionic variables are replaced by the operators $(-\mathrm{i}\hbar\delta/\delta\eta_\mu)$ and $(-\mathrm{i}\hbar\delta/\delta\eta^{*\nu})$ and is explicitly given by

$$\begin{aligned}\left(\frac{\delta S}{\delta\psi^{*\mu}(t)}\right)_{\mathrm{op}}=&\ \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\mu}(t)}\right)-\mathrm{i}\Gamma^{\nu}_{\mu\alpha}[q(t)]\dot q^{\alpha}(t)\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\nu}(t)}\right)\\&-\tfrac{1}{2}R_{\mu\gamma}{}^{\beta\delta}[q(t)]\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta_\gamma(t)}\right)\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\beta}(t)}\right)\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\delta}(t)}\right).\end{aligned} \tag{B2}$$

Its solution is the two-point fermionic generating functional

$$Z^{(2)}_\sigma[\eta^*,\eta]=Z_\sigma[\eta^*,\eta]|_{\eta^*=\eta=0}\exp\left(-\frac{1}{\hbar}\int\mathrm{d}t\,\mathrm{d}t'\,\eta^{*\mu}(t)G^{\nu}_{\mu}(t-t')\eta_\nu(t')\right) \tag{B3}$$

where the fermionic propagator is given by

$$G^{\nu}_{\mu}(t-t')=\theta(t-t')T\left[\exp\int_{t'}^{t}\mathrm{d}t_1\left(\Gamma^{\nu}_{\mu\alpha}[q(t_1)]\dot q^{\alpha}(t_1)-\frac{\mathrm{i}\hbar}{2}R^{\mu}_{\nu}[q(t_1)]\right)\right] \tag{B4}$$

and $Z^{(2)}_\sigma[\eta^*,\eta]|_{\eta^*=\eta=0}$ is given in (3.3).

The second recurrence relation for the Green functions is

$$\begin{aligned}&\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta^{*\omega}(t_2)}\right)\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta_\sigma(t_1)}\right)\left(-\mathrm{i}\hbar\frac{\delta}{\delta\eta_\tau(t_0)}\right)\left[\left(\frac{\delta S}{\delta\psi^{*\mu}(t)}\right)_{\mathrm{op}}+\eta^{*\mu}(t)\right]\\&\qquad\times Z_\sigma[\eta^*,\eta]|_{\eta^*=\eta=0}=0.\end{aligned} \tag{B5}$$

The solution of this Green function equation will contain a four-point fermionic general functional. The solution of the Green function equation (B5) is the required fermionic generating functional that can be used to obtain the kernel for the Schrödinger equation. The generating functional has the form

$$Z_\sigma[\eta^*,\eta]=Z^{(2)}_\sigma[\eta^*,\eta]Z^{(4)}_\sigma[\eta^*,\eta] \tag{B6}$$

where the four-point functional is

$$Z_{\sigma}^{(4)}[\eta^*, \eta] = \exp\left(-\frac{1}{4\hbar} \int \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}t_3\, \mathrm{d}t_4\, \eta^{*\mu}(t_1) G_{\mu}^{\alpha}(t_1 - t) \eta^{*\nu}(t_2) G_{\nu}^{\gamma}(t_2 - t) \right.$$
$$\left. \times D_{\alpha\gamma}{}^{\beta\delta}[q(t), q(t')] G_{\beta}^{\omega}(t' - t_3) \eta_{\omega}(t_3) G_{\delta}^{\sigma}(t' - t_4) \eta_{\sigma}(t_4) \right). \tag{B7}$$

In (B7) the four-fermion vertex is given by

$$D_{\alpha\gamma}{}^{\beta\delta}[q(t), q(t')]$$
$$= \delta(t - t') R_{\alpha\gamma}{}^{\beta\delta}[q(t)] + \frac{\mathrm{i}\hbar}{2} R_{\alpha\gamma}{}^{bd}[q(t)] G_b^a(t - t') G_d^c(t - t') R_{ac}{}^{\beta\delta}[q(t')]$$
$$+ \int \mathrm{d}t'' \frac{\mathrm{i}\hbar}{2} R_{\alpha\gamma}{}^{bd}[q(t)] G_b^a(t - t'') G_d^c(t - t'') \frac{\mathrm{i}\hbar}{2} R_{ac}{}^{b'd'}[q(t'')]$$
$$\times G_{b'}^{a'}(t'' - t') G_{d'}^{c'}(t'' - t') R_{a'c'}{}^{\beta\delta}[q(t')] + \ldots . \tag{B8}$$

This vertex is shown diagrammatically in figure 2. It should be noted that the process of solving the recurrence relations for the Green functions can be carried out iteratively to obtain the N-point fermionic generating functional. Hence the formal solution of the Dyson–Schwinger equation would be the infinite product $Z_{\sigma}[\eta^*, \eta] = \prod_{j=1}^{\infty} Z_{\sigma}^{(2j)}[\eta^*, \eta]$.

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